# An Efficient and Robust Design for Transshipment Networks

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Transshipment, the sharing of inventory among parties at the same echelon level of a supply chain, can be used to reduce costs. The effectiveness of transshipment is in part determined by the configuration of the transshipment network. We introduce chain configurations in transshipment settings, where every party is linked in one connected loop. Under simplifying assumptions we show analytically that the chain configuration is superior to configurations suggested in the literature. In addition, we demonstrate the efficiency and robustness of chain configurations for more general scenarios and provide managerial insights regarding preferred configurations for different problem parameters.

Key Words: Inventory Transshipment, Network Design

# 1 Introduction

According to the 15th Annual State of Logistics Report (Council of Logistics Management (CLM), 2004), logistic costs in the United States have risen, from $910 billion in 2002 to $936 billion in 2003. Inventory costs account for a third of this total. In the report, "the ability to respond faster to changing customer needs" and "the flexibility to adjust manufacturing and delivery cycles" are identified as keys to success in this competitive environment. Even industries with stable demand patterns spend millions of dollars each year coping with uncertainty in customer demand and operating costs. Uncertainty can lead to major supply chain inefficiencies, causing lost revenue, poor customer service, high inventory levels and unrealized profits.

Inventory transshipment is a promising strategy to provide operational flexibility to mitigate the effects of demand uncertainty. Transshipment is the sharing of inventory among locations at the same echelon level of a supply chain. For example, in Figure 1, four retailers are supplied from one warehouse. Rather than relying solely on their own inventory or costly emergency replenishment from the warehouse, retailers can collaborate to address demand uncertainty. In cases where safety stock is held, transshipment achieves the benefits of risk

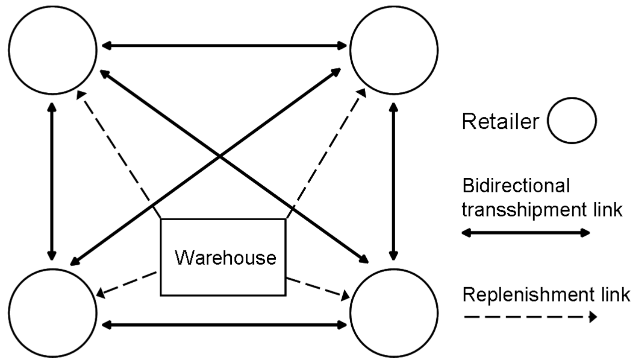



Figure 1: Supply Chain With Transshipment

pooling to meet uncertain demand while reducing inventory levels at individual locations, see Dong and Rudi (2004). Companies using transshipments include Footlocker, Macy's, and a group of chip manufacturers (NEC, Toshiba) sharing a common supplier, ASML (Yücesan, 2005).

Tagaras (1999) and Herer et al. (2002) considered restrictions on transshipment network connectivity when a complete network (i.e., locations may transship directly to all other locations) is not possible. They study networks where locations are divided into groups and transshipment is allowed only within groups. In this paper, we term these networks as *group* configurations. Figure 2(a) is an example of a group configuration with group sizes of two and Figure 2(b) shows a group configuration with group sizes of three. A complete network is a group configuration where all locations belong to one group. Tagaras (1999) and Herer et al. (2002) show that while group configurations cannot achieve all of the savings of a complete network, the savings are considerable, and the number of links is smaller. Establishing a link between locations requires investments in bidirectional communication channels, physical distribution systems and financial and administrative arrangements. Transshipment networks without direct links between all locations tend to consolidate transshipment flows on a few routes, which can reduce the demand for communication channels and transportation (vehicles and drivers), and lower the overall complexity of a system. This is particularly important in the case of outsourced transportation between locations that is negotiated in advance or in the case of multiple products which share common transshipment methods.

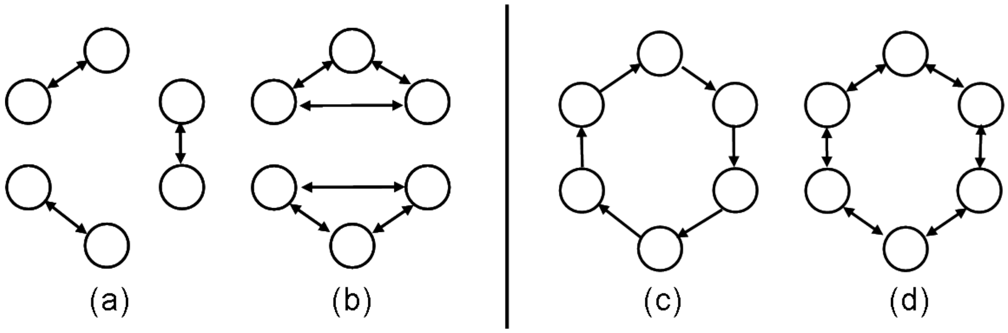



Figure 2: Group and chain configurations: (a) groups of two, (b) groups of three, (c) unidirectional chain, (d) bidirectional chain

In a *chain* configuration, every location is connected to two locations forming a continuous loop. In an *unidirectional* chain network, see Figure 2(c), a location can transship to one neighboring location and receive from another. In a *bidirectional* chain network, see Figure 2(d), locations can transship and receive from both neighboring locations. In this paper, we demonstrate the cost efficiency of the chain configurations both analytically and numerically. Under simplifying assumptions, we prove that the chain configuration is more cost efficient than certain group configurations that appear in the literature. Through an extensive numerical study, we demonstrate the efficiency and robustness of the chain configuration in more general settings, as compared with other possible configurations.

The paper is organized as follows. In Section 2, we present relevant literature on transshipment problems and chain configurations in various applications. In Section 3, we describe operational and strategic transshipment problems. In Section 4, we analytically compare the chain with group configurations and, in Section 5, we present numerical results comparing the chain with other configurations in more general settings. We conclude with a discussion of research extensions in Section 6.

# 2 Literature review

The transshipment literature has been focused on operational decisions for a fixed network design: the transshipment amount between locations and the replenishment amount from the supplier at each location. Most authors consider two locations, see Tagaras (1989), Tagaras and Cohen (1992) and Tagaras and Vlachos (2002); or locations that are identical in cost parameters, see Krishnan and Rao (1965) and Tagaras (1999). Robinson (1990) and more recently Herer et al. (2006) consider locations that vary by demand distributions and cost parameters. Tagaras and Cohen (1992) and Tagaras and Vlachos (2002) allow the replenishment lead time to be larger than one; in others, transshipment lead time is negligible and replenishment lead time is one period.

Research on transshipment networks other than group configurations is more recent. Herer et al. (2006) compare five transshipment configurations that differ in their transshipment capabilities (number and cost of links) and demonstrate the value of transshipments in each. In addition, the authors construct a sample-path-based optimization procedure to calculate retailers' optimal order quantities. In parallel to our work, Yu et al. (2005) consider a transshipment network with one supplier and three retailers, and study six network design possibilities which they refer to as operational flexibility levels. They apply the newsvendor network model of Van Mieghem and Rudi (2002) to find the retailers' optimal order quantities for any given flexibility level and analyze the interaction between optimizing order quantities and increasing operational flexibility.

While the transshipment literature has been on only a limited number of network configurations, other network configurations have been considered in the manufacturing and service operations literature. The chain configuration has been shown to be an efficient structure in production. Jordan and Graves (1995) and Graves and Tomlin (2003) show that if plants are assigned in a chain-like manner to produce two types of products, one can achieve most of the potential benefit of complete flexibility, in which all plants are able to produce all products. Others such as Sheikhzadeh et al. (1998), Gurumurthi and Benjaafar (2001), Hopp et al. (2004), Iravani et al. (2005, 2007a, 2007b), Jordan et al. (2004) and Chou et al. (2005) highlight the properties of the chain structure in different production and maintenance environments.

# 3 Problem and Model Description

We first present the model and assumptions concerning the transshipment and replenishment mechanisms. We review the operational transshipment problem from the literature in Section 3.1 and introduce the strategic transshipment network design problem in Section 3.2. Given $N$ retailers, facing stationary random demand, events occur as follows in each period:

1. Replenishment from the warehouse arrives from orders made in the previous period; backlogged demand is satisfied. The inventory level is equal to the order-up-to level.
2. Demand is observed.
3. Transshipment decisions are made and occur immediately. Transshipment costs are incurred.
4. Demand is satisfied or backlogged. Holding and shortage costs are incurred.
5. Inventory level is updated.
6. Replenishment orders are made according to an order-up-to policy.

We assume that the replenishment lead time from the warehouse is one time period, and that the warehouse has sufficient capacity to respond to all orders. Transshipment lead times are negligible; hence transshipments serve as a quicker, alternative supply if demand exceeds available inventory. Herer et al. (2006) prove that an order-up-to replenishment policy, together with an optimal transshipment policy, minimizes the expected per period inventory holding, shortage and transshipment costs. Using an order-up-to policy, the system regenerates every period; therefore, minimizing the expected cost for one period is equivalent to minimizing the long-run expected costs.

## 3.1 Existing work on operational transshipment problems

The objective of the operational transshipment problem is to minimize the expected cost per period for a given network design. The following parameters are used:

| | |
|---|---|
| $\mathcal{N}$ | set of retail locations (also called "nodes"), $i \in \{1, ..., N\}$ |
| $\mathcal{K}$ | set of directed transshipment links $(i, j) \in \mathcal{K}$, defined by configuration $\mathcal{K} \subseteq (\mathcal{N} \times \mathcal{N})$ |
| $D_i$ | random variable denoting the demand at location $i \in \mathcal{N}$ in a period |
| $c_t$ | cost of transshipping one unit along one link |
| $c_s$ | cost of one unit of shortage for one period |
| $c_h$ | cost of holding one unit in inventory for one period. |

The following are decision variables:

| | |
|---|---|
| $S_i$ | order-up-to level at location $i \in \mathcal{N}$ |
| $X_{ij}$ | number of items to transship on link $(i, j) \in \mathcal{K}$ |
| $I_i^+$ | net surplus at end of time period (after transshipment) at location $i \in \mathcal{N}$ |
| $I_i^-$ | net shortage at end of time period (after transshipment) at location $i \in \mathcal{N}$. |

When locations follow an order-up-to policy, the total replenishment of the system is equal to the total demand observed. Since variable replenishment costs are the same across locations, they do not affect the transshipment decisions and can be omitted from our model. In addition, we assume that the fixed costs of replenishment and transshipment are incurred every period.

We first discuss optimizing $\mathbf{X}$, the matrix of $X_{ij}$, for a general order-up-to level, and then present an iterative method to find the optimal order-up-to levels. In each period, for a given order-up-to level vector $\mathbf{S}$ and an observed demand vector $\mathbf{d}$, the following linear program is solved to determine transshipment flows.

$$z(\mathcal{K}, \mathbf{S}, \mathbf{d}) = \min_{\mathbf{X}} c_t \sum_{(i,j)\in\mathcal{K}} X_{ij} + c_s \sum_{i\in\mathcal{N}} I_i^- + c_h \sum_{i\in\mathcal{N}} I_i^+ \tag{1a}$$

subject to

$$\sum_{j:(i,j)\in\mathcal{K}} X_{ij} - \sum_{j:(j,i)\in\mathcal{K}} X_{ji} + I_i^+ - I_i^- = S_i - d_i \qquad \forall i \in \mathcal{N} \tag{1b}$$

$$\sum_{j:(i,j)\in\mathcal{K}} X_{ij} \leq S_i \qquad \forall i \in \mathcal{N} \tag{1c}$$

$$X_{ij} \geq 0 \qquad \forall (i,j) \in \mathcal{K} \tag{1d}$$

$$I_i^+, I_i^- \geq 0 \qquad \forall i \in \mathcal{N} \tag{1e}$$

The objective function (1a) is to minimize the sum of transshipment costs, shortage costs and holding costs, given $\mathcal{K}$, $\mathbf{S}$ and $\mathbf{d}$. Constraints (1b) are for requiring that demand must be satisfied from inventory and/or transshipments, or backlogged. Constraints (1c) are for limiting transshipment amounts by the order-up-to level. We assume that transshipped units may only traverse one link in a period; however, up to its order-up-to quantity, node $i \in \mathcal{N}$ may both receive units from location $j_1$ and transship units to location $j_2$ for some $j_1, j_2$ such that $(j_1, i) \in \mathcal{K}$ and $(i, j_2) \in \mathcal{K}$. This would occur if capacity is shifted among locations where $j_1$ has a surplus and $j_2$ has a shortage. Lastly, constraints (1d) and (1e) are non-negativity constraints for transshipments, inventory and shortage levels.

To find the optimal order-up-to levels and the optimal expected cost for a given configuration, Herer et al. (2006) suggest an infinitesimal perturbation analysis (IPA) procedure. IPA is a sample-path optimization technique, in which the gradients of the expected total cost with respect to order-up-to levels are estimated by solving formulation (1a)-(1e) for different demand realizations and for candidate vector $\mathbf{S}$. The gradient values are used to update $\mathbf{S}$, and the procedure is guaranteed to converge to the optimal order-up-to vector, $\mathbf{S}^{\mathcal{K}}$, for configuration $\mathcal{K}$. The operational transshipment problem is then solved for random demand observations to calculate the optimal expected cost.

## 3.2 The strategic transshipment network design problem

The objective of the strategic transshipment network design problem is to determine the optimal network configuration (i.e., the link set $\mathcal{K}$) given a limit, $P$, on the number of allowable transshipment links. To compare the efficiency of different transshipment networks, we introduce $Z(\mathcal{K})$ to denote the optimal *expected* cost (over demand realizations) of a network with transshipment link set $\mathcal{K} \subseteq (\mathcal{N} \times \mathcal{N})$; i.e., $Z(\mathcal{K}) = E_{\mathbf{D}}[z(\mathcal{K}, \mathbf{S}^{\mathcal{K}}, \mathbf{d})]$. The transshipment network design problem is formulated:

$$\min_{\mathcal{K}} Z(\mathcal{K}) \quad \text{subject to} \quad |\mathcal{K}| \leq P \tag{2}$$

The objective is for minimizing the expected cost subject to the limit on the size of the link set. Efficient networks are those that have low values of $Z(\mathcal{K})$.

# 4 Analytical Results

In this section, we analytically compare the performance of the unidirectional chain and the bidirectional chain to group configurations with the same number of links. Let group($\ell$) denote a group configuration of $M$ groups of $\ell$ nodes and $\ell(\ell-1)$ directed links each, for any positive integer $M$ such that $\ell M = N$, where $N$ is the number of nodes in the problem. Group(2) has the same number of links as the unidirectional chain, and group(3) has the same number of links as the bidirectional chain. We show that the group configuration incurs higher expected costs than the chain configuration under the simplifying assumptions described below.

## 4.1 Chain and Group Transshipment Networks

A transshipment over one or more links is *cost-reducing* if the incurred transshipment costs are less than the sum of holding and shortage costs. For example, if $2c_t \leq c_h + c_s$, a transshipment over two links is cost-reducing. We define a *shift* to be a transshipment along a single link. Accordingly, the number of *profitable shifts* is the maximum number of links along which units may be transferred to meet shortage and still reduce costs. For example, in the case where $c_h = 2$, $c_s = 11$ and $c_t = 6$, the number of profitable shifts is 2.

To obtain our results, we assume that the locations are identical in their costs and demand distributions, and face independent demand. This is appropriate in cases of retailers serving near-homogeneous populations over moderately sized geographic regions. Further, beginning with an identical cost and location model allows us to develop analytical results and gain basic insights on preferred networks for settings with non-identical costs and locations. We investigate more general cases numerically in Section 5.

In Theorem 1, we prove that the unidirectional chain outperforms group(2), with $N = 2M$

directed links. We first present Lemmas 1 and 2 which are needed in the proof of Theorem 1. All proofs are presented in Appendix A.

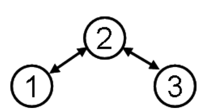
**Lemma 1.** *Given a fixed configuration with cost parameters that allow for only one profitable shift: (1) minimizing the expected number of units in inventory, (2) minimizing the expected number of units in shortage or (3) maximizing the number of cost-reducing transshipments will yield the minimum total expected cost.*

**Lemma 2.** *In a chain or group($\ell$) configuration, the optimal order-up-to levels are identical across locations.*

**Theorem 1.** *Given a network with $N = 2M$ locations ($M = 1, 2, \cdots$), the optimal expected cost in a unidirectional chain configuration is less than or equal to the optimal expected cost in a group(2) configuration.*

In Theorem 3, we claim that the bidirectional chain outperforms group(3) in expectation. For ease of presentation, we first prove the claim for a six-node network in Theorem 2. We begin by presenting the lemmas required to prove the theorems.

**Lemma 3.** *Consider three nodes linked in the following manner:*




*and assume their order-up-to levels are identical and the cost parameters allow for only one profitable shift. The ending inventory/shortage after transshipments at nodes 1 and 3 are positively (in the weak sense) correlated.*

**Lemma 4.** *The optimal transshipment quantities for a group configuration of three nodes may be obtained in two steps, where in the first step optimal transshipments are made along two links only, and in the second step units are optimally transshipped along the third link, without changing the transshipment quantities in the first step.*

**Theorem 2.** *Given a network with six locations, the optimal expected cost in a bidirectional chain configuration is less than or equal to the optimal expected cost in a group(3) configuration.*

**Theorem 3.** *Given a network with $N = 3M$ locations ($M = 1, 2, \cdots$), the optimal expected cost in a bidirectional chain configuration is less than or equal to the optimal expected cost in a group(3) configuration.*

From the proofs, we see that for one profitable shift, the expected costs of the unidirectional chain and group(2) are identical. For more than one profitable shift, the unidirectional chain is more efficient than group(2) because the chain configuration allows for transshipment across multiple links. Further, the bidirectional chain is more efficient than group(3) in the case of one profitable shift. The value of the bidirectional chain configuration does not hinge only on the ability to shift capacity along multiple links; but as the proof of Theorem 2 illustrates, it is also based on the ability to draw inventory from diverse (loosely correlated or uncorrelated) sources.

## 4.2 Link to Manufacturing Process Flexibility

Note that formulation (1a)-(1e) with $c_s = 1$ and $c_h = c_t = 0$ is equivalent to the manufacturing process flexibility model in Jordan and Graves (1995), where the objective is to minimize shortage. In the manufacturing setting, plants satisfy demand of other plants by shifting capacity along multiple links with no penalty. In the transshipment setting, positive holding and transshipment costs limit this flexibility. In Jordan and Graves (1995), plant capacity is independent of network design; in transshipment, retailers adjust order-up-to levels to minimize system costs.

Our claims remain valid for arbitrary order-up-to levels as long as they are identical across locations; therefore, the theorems presented in a transshipment context are applicable in the manufacturing process flexibility setting. The unidirectional chain is analogous to the manufacturing chain assignment in Jordan and Graves (1995). Consider the six node example shown in Figure 3 where (a) is the unidirectional chain transshipment network and (b) is a six plant and six product manufacturing assignment. Each plant's capacity is associated with the order-up-to level of a retailer with the same number. Also, each product's demand is associated with a retailer's demand. In the manufacturing flexibility figure, a solid arrow represents an assignment of a product to a plant with the same number and a dashed arrow represents an assignment of a product to a plant with a different number. In transshipment setting terms, a solid arrow represents a retailer satisfying their demand with their own inventory, while a dashed arrow represents a transshipment link.

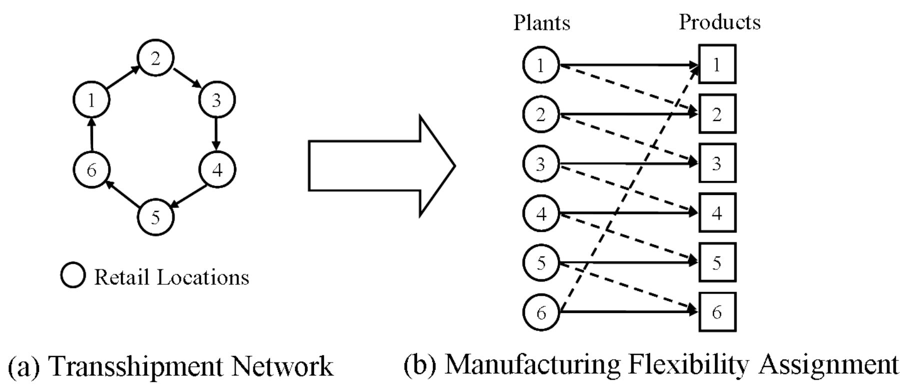



Figure 3: Transshipment unidirectional chain and manufacturing chain assignment networks

Similarly the bidirectional chain is analogous to a manufacturing assignment where each plant $i$ is assigned products $i-1$, $i$ and $i+1$. Consider the six node example shown in Figure 4 where (a) is the bidirectional chain network and (b) is a six plant and six product manufacturing assignment. Compared to Figure 3(b), each plant in Figure 4(b) has the flexibility to make an additional product.

Our theorems are applicable to manufacturing flexibility assignment scenarios where the number of plants and products are equal, and capacity at each plant is identical. In the manufacturing setting, Theorem 1 states that assigning flexibility to create one large chain is better than creating several smaller chains of two plants and two products each. Theorems 2 and 3 claim the same principle in the case where each plant is able to manufacture three products.

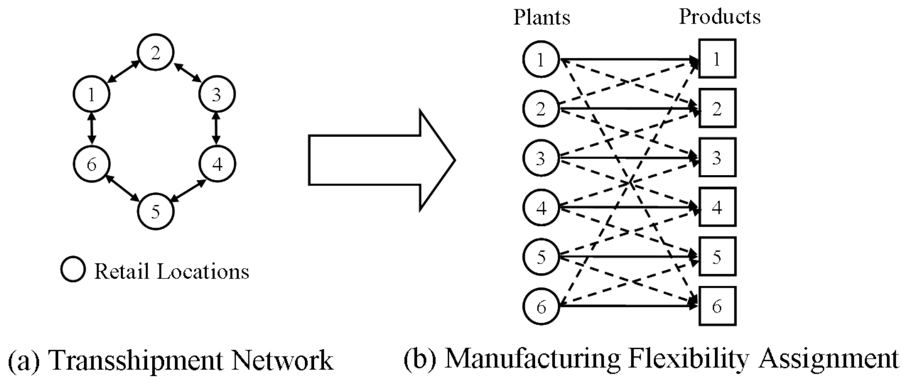



Figure 4: Transshipment bidirectional chain and corresponding manufacturing assignment

# 5 Numerical results

We extend our analysis to other configurations and settings with non-identical demand parameters, non-identical transshipment costs and correlated demand. We focus on configurations with bidirectional transshipment links. In Section 5.1 we present our study on the efficiency of the chain network, and in Section 5.2 we present our study on the robustness of the chain network.

## 5.1 Efficiency of chain configurations

We compare the bidirectional chain with other configurations with $N$ bidirectional links. Three network sizes are included in the experiments: $N = 6, 12, 18$. We fix the sum of holding and shortage costs at 13 with three pairs of holding and shortage costs, denoted by $\tau = (c_h, c_s) \in \{(2, 11), (4, 9), (6, 7)\}$, and consider six transshipment costs: $c_t = 2, 4, 6, 8, 10, 12$. Demand at each node follows a Gamma distribution with mean of 100 and five coefficients of variation: $\gamma = 0.25, 0.5, 1, 1.5, 2$. For each scenario, we perform pairwise comparisons of the chain with the other configurations.

Consider a pairwise comparison of the chain with configuration $f$. Let $Z(\mathcal{K}_f)$ be the optimal expected cost of configuration $f$. We compute the percent of pairwise comparisons in which the chain has a lower expected cost than configuration $f$ with scenario values $\gamma$, $c_t$, and $\tau$. In pairwise comparisons in which the chain is not the lower cost configuration, let $\Delta_f(\gamma, c_t, \tau)$ denote the deviation of the cost of the chain from the cost of configuration $f$:

$$\Delta_f(\gamma, c_t, \tau) = \frac{Z(\mathcal{K}_c) - Z(\mathcal{K}_f)}{Z(\mathcal{K}_f)} \times 100\%.$$

We also compute the average deviation of all pairwise comparisons in which the chain does not yield the lower cost and the maximum deviation of these comparisons. Further, we compare the relative magnitude of $\Delta_f(\gamma, c_t, \tau)$ with the maximum difference between the expected costs for all configurations for scenario values $\gamma$, $c_t$, and $\tau$.

For networks with $N = 6$, there are 21 possible configurations, shown in Appendix B. In Table 1, we present the results as a function of the coefficient of variation, which is found to have the most significant impact on the results.

| $\gamma$ | # of pairwise comparisons | Chain is most efficient(%) | $\Delta_f(\gamma, \cdot, \cdot)$ Avg. | Max. | Maximum difference |
|---|---|---|---|---|---|
| 0.25 | 360 | 100% | - | - | 38.2% |
| 0.5 | 360 | 100% | - | - | 38.3% |
| 1 | 360 | 100% | - | - | 35.3% |
| 1.5 | 360 | 98.3% | 1.2% | 2.6% | 27.2% |
| 2 | 360 | 90.0% | 1.6% | 7.5% | 20.1% |
| Total | 1800 | 97.7% | 1.4 % | 7.5% | 38.3% |

Table 1: Efficiency of the chain network for $N = 6$ as a function of $\gamma$

The chain is more efficient in 97.7% of all comparisons. The chain is the most efficient configuration for values of $\gamma \leq 1$, implying that the chain remains efficient regardless of the cost parameters for low to moderate levels of demand uncertainty. At $\gamma = 2$, other configurations may be more efficient than the chain, although the chain is more efficient in 90% of the comparisons. The maximum deviation between the chain and more efficient configurations is 7.5%, which is small relative to the maximum difference of 20%. The maximum difference among all configurations for each scenario decreases as $\gamma$ increases, which is counterintuitive since the number of transshipments should increase with $\gamma$ and therefore the network design should be more important. This can be explained by observing Figure 5 in which, a plot of the costs of the chain, network 5 and network 19 for $c_h = 2, c_s = 11$

and $c_t = 2$ are drawn. We see that the cost of each network rises significantly as $\gamma$ increases. Further, the absolute difference between the networks increases with $\gamma$. Thus in terms of

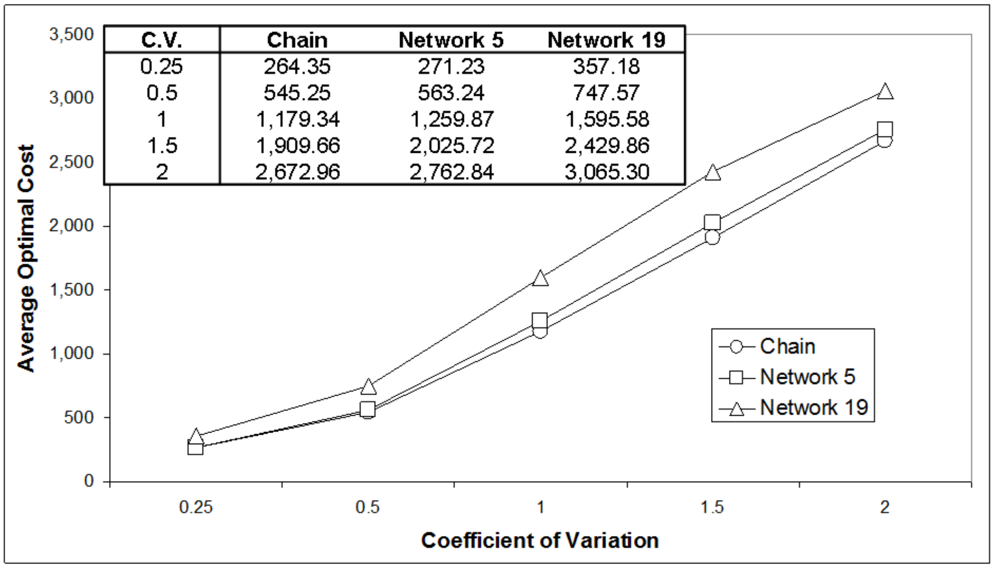



Figure 5: Average optimal costs as a function of $\gamma$

absolute costs, the importance of network design increases with $\gamma$. The maximum difference values in Table 1 are relative, thus they are decreasing with $\gamma$ as the cost of each network rises significantly when $\gamma$ increases.

In Tables 2 and 3, we explore the efficiency of the chain with respect to cost parameters for $\gamma$=2 (i.e., the fifth row of Table 1). In Table 2, the results are presented as a function of holding and shortage costs, and in Table 3, results are presented as a function of transshipment costs.

| $c_h$ | $c_s$ | # of pairwise comparisons | Chain is most efficient(%) | $\Delta_f(2,\cdot,\tau)$ Avg. | $\Delta_f(2,\cdot,\tau)$ Max. | Maximum difference |
|---|---|---|---|---|---|---|
| 2 | 11 | 120 | 95.0% | 1.2% | 4.8% | 20.1% |
| 4 | 9 | 120 | 90.0% | 1.9% | 7.5% | 13.9% |
| 6 | 7 | 120 | 85.0% | 1.5% | 7.0% | 8.7% |
| Total | | 360 | 90.0% | 1.6% | 7.5% | 20.1% |

Table 2: Efficiency of the chain network for $N = 6$ for $\gamma$= 2 as a function of $c_h$ and $c_s$

In Table 2, we observe that chain is again more efficient in most cases. The percent of cases in which chain is most efficient as well as the maximum deviation among all configurations increases with the critical ratio $\frac{c_s}{c_s+c_h}$. For higher critical ratios, the optimal order-up-to levels are typically higher, which combined with high demand uncertainty ($\gamma = 2$), allows for more transshipments. Thus network design becomes more important and the chain configuration allows for more transshipment possibilities. Further, the cost of a transshipment network does not change significantly for different critical ratios; the maximum deviation in Table 2 is representative of the absolute cost difference between the worst and best configurations.

| $c_t$ | # of pairwise comparisons | Chain is most efficient(%) | $\Delta_f(2, c_t, \cdot)$ Avg. | $\Delta_f(2, c_t, \cdot)$ Max. | Maximum difference |
|---|---|---|---|---|---|
| 12 | 60 | 100% | - | - | 0.8% |
| 10 | 60 | 100% | - | - | 2.4% |
| 8 | 60 | 100% | - | - | 4.4% |
| 6 | 60 | 98.3% | 0.01% | 0.01% | 7.2 % |
| 4 | 60 | 78.3% | 0.7% | 2.1% | 12.6 % |
| 2 | 60 | 63.3% | 2.3% | 7.5% | 20.1 % |
| Total | 360 | 90.0% | 1.6% | 7.5% | 20.1% |

Table 3: Efficiency of the chain network for $N = 6$ and $\gamma = 2$ as a function of $c_t$

In Table 3, the cases in which the chain is less efficient correspond to scenarios with high demand variability and low transshipment costs (i.e., $\gamma = 2$ and $c_t = 2$ and 4). The maximum difference increases as $c_t$ decreases; with low $c_t$, the transshipment quantities are expected to increase, in which case the network design is more important. In addition, when demand variability is high and transshipment costs are low relative to holding and shortage costs, configurations that centralize inventory to maximize pooling are desirable. In these cases, the most efficient configuration is network 4 in Appendix B, which we refer to as the *star* network (an extra link exists compared to a typical star since all configurations must include 6 links). The star network stores additional inventory at one centralized node and transships items to other nodes as needed. This results in a lower cost than the chain since in the star network, transshipments are made along a single link, which is important when demand is highly variable.

**Observation 1.** *When demand uncertainty is low or transshipment costs are high relative to holding and shortage costs, configurations that utilize multiple shifts are efficient. Under* *these conditions, balanced order-up-to levels among all locations are desirable and the chain is likely to be the most efficient configuration for N=6.*

**Observation 2.** *When demand uncertainty is high and transshipment costs are low relative to holding and shortage costs, configurations that utilize multiple shifts are not the most efficient. Rather, these conditions promote centralized risk pooling and the star configuration is likely to be the most efficient configuration for N=6.*

In numerical experiments we confirm that Observations 1 and 2 hold for networks with $N = 12$ and 18. Since the number of possible unique configurations for 12 and 18 location/link scenarios are much greater (e.g., over 400 configurations for $N = 12$), we consider 25 randomly generated unique networks in the pairwise comparisons with the chain and star configurations. The results are presented in Appendix C. The chain is more efficient in over 97% of the comparisons with the randomly generated configurations. When the chain is inferior, the average value of $\Delta$ is less than 2% with a maximum deviation below 7%.

Average costs of the chain and star transshipment networks for a specific setting are presented in Table 4, where $Z(\mathcal{K}_s)$ represents the cost of a star network. Consistent with Observation 2, the star performs significantly better (26%) than the chain when $c_t = 2$ is small relative to $c_s + c_h$ and $\gamma = 2$, while in the other cases the chain is more efficient.

| | Transshipment Cost | |
|---|---|---|
| $\gamma$ | 2 | 6 |
| 0.5 | $Z(\mathcal{K}_c) = 1,501.2, Z(\mathcal{K}_s) = 1,610.2$ | $Z(\mathcal{K}_c) = 2,388.8, Z(\mathcal{K}_s) = 2,571.1$ |
| 2 | $Z(\mathcal{K}_c) = 7,872.8, Z(\mathcal{K}_s) = 6,255.8$ | $Z(\mathcal{K}_c) = 10,081.3, Z(\mathcal{K}_s) = 10,115.31$ |

Table 4: Chain and Star Network Costs for $N = 18, c_h = 2, c_s = 11$

In a direct comparison, for cases in which the star is more efficient than the chain, the difference is as much as 7.5%, 21% and 26% for 6, 12 and 18 node scenarios, respectively. The chain, however, is more efficient in over 80% of the scenarios we studied. In these cases, the chain transshipment network is as much as 18% more efficient than the star.

## 5.2 Robustness of chain configurations

In the following studies, we relax the assumptions of homogeneous transshipment costs, identical locations and independent demand. Note that, a 6 retailer - 6 link problem with unique retailers may have as many as 720 unique *linkings*. We define a linking as an arrangement of retailers in a given configuration. Obtaining the optimal expected cost of each linking requires up to two minutes of CPU time on a Sun Fire v250 1.28-GHz UltraSPARC IIIi computer with dual processors. To maintain manageable simulation times, we limit our study to 6 retailer - 6 link problems. For each configuration, we refer to the linking which yields the minimum expected cost as its *best linking* and the linking with the highest expected cost as the *worst linking*. In Section 5.2.1 we study different transshipment costs and in Section 5.2.2 we study different demand distributions. Lastly, in Section 5.2.3 we study correlated demand.

### 5.2.1 Nonidentical Transshipment Costs

We consider a model in which retailers are divided into two sets. Let $c_t^l$ be the transshipment cost between retailers in the same set and $c_t^h$ the cost between retailers in the different sets; $c_t^l$ represents low transshipment costs and $c_t^h$ represents high transshipment costs, so $c_t^h > c_t^l$. Thus, we study transshipment network design in scenarios where the cost of transshipping between certain groups of retailers is costly and undesirable.

In the numerical experiment, we consider two demand scenarios where $\gamma = 0.5$ and 1.5, with costs $c_h = 2$ and $c_s = 11$. Several combinations of $c_t^l$ and $c_t^h$ are included in this experiment. In Table 5, we present the optimality gap results for the best and worst linkings of the chain, star and group networks for the case where the retailers are divided into two sets of three retailers each and for $\gamma = 1.5$. The results for $\gamma = 0.5$ are similar.

For low transshipment costs (lines 2 and 3 in the Table 5), a properly linked chain configuration is optimal. For high transshipment costs (lines 4 and 5 in Table 5), a properly linked group configuration is optimal. This follows intuitively, since transshipping between

| $(c_t^l, c_t^h)$ | Linking Optimality Gap | | | | | |
|---|---|---|---|---|---|---|
| | Best chain | Worst chain | Best star | Worst star | Best group | Worst group |
| (2,3) | 0.0% | 6.5% | 2.3% | 4.1% | 10.5% | 13.7% |
| (2,4) | 0.0% | 12.5% | 3.9% | 7.3% | 8.0% | 14.3% |
| (6,9) | 1.8% | 12.4% | 6.3% | 9.6% | 0.0% | 7.1% |
| (6,12) | 5.0% | 25.9% | 10.0% | 16.1% | 0.0% | 14.2% |

Table 5: Two sets of three retailers each, $\gamma = 1.5$

retailers in different sets is costly. For the chain and group networks, the arrangement of the retailers in the configuration has significant impact. For the chain, it is optimal to maximize the links between retailers of the same set. For the group configuration, it is optimal to arrange retailers from different sets into different groups, so that all links have low transshipment cost. In contrast, the efficiency of the star network is not as sensitive to retailer arrangement except in the case of highest transshipment costs. In all star network linkings, retailers will be reliant on transshipments via the higher cost links.

In additional studies of other sets (1 retailer and 5 retailers; 2 retailers and 4 retailers) the results are similar; the efficiency of the chain configuration improves with lower transshipment costs and the efficiency of the group configuration improves for higher transshipment costs. The cost of a properly arranged chain configuration is within 5% of the cost of the optimal network in all experiments. We conclude that the chain configuration is an efficient transshipment network in this nonidentical trannshipment cost setting and its efficiency is robust to the differences in transshipment costs.

### 5.2.2 Nonidentical Demand Distributions

We consider nonidentical demand means with ranges of 100 units and 200 units. Mean demands are distributed evenly throughout the range among the six retailers. For example, for a range of 100, the mean demands at the retailers are 100, 120, 140, 160, 180 and 200. The coefficient of variation at each retailer is fixed at 0.5 or 1.5 depending on the scenario, so that demand standard deviations vary among retailers accordingly. We consider scenarios with $c_t = 2$ and 6, and with $c_h = 2$ and $c_s = 11$.

The results are summarized in Table 6. The demand parameters and transshipment cost of the scenario are listed in the first three columns. In the fourth column is the optimality gap between the best chain linking and the best linking for all configurations for that scenario. The same gap for the worst chain linking is presented in the fifth column. In the sixth column, we report the *spread*, which is the gap between the lowest and highest best linking values of all configurations. In this numerical study, the spread measures the maximum impact of transshipment network design; in addition, it is useful for evaluating the magnitude of the other measures. In the last two columns, we illustrate the importance of linking for a transshipment configuration. *Chain linking difference* represents the difference between the worst chain linking and the best chain linking. *Largest linking difference* reports the maximum gap between the best and worst linking for all configurations.

| Scenario | | | Best chain | Worst chain | | Chain linking | Largest linking |
|---|---|---|---|---|---|---|---|
| $\mu$ | $\gamma$ | $c_t$ | linking gap | linking gap | Spread | difference | difference |
| 100-200 | 0.5 | 2 | 0% | 1.5% | 29% | 1.1% | 11.7% |
| 100-200 | 1.5 | 2 | 3% | 0.1% | 23% | 2.3% | 15.1% |
| 100-200 | 0.5 | 6 | 0% | 1.5% | 7% | 1.2% | 7.2% |
| 100-200 | 1.5 | 6 | 1% | 2.2% | 6% | 1.6% | 8.7% |
| 100-300 | 0.5 | 2 | 1% | 3.5% | 24% | 2.8% | 18.5% |
| 100-300 | 1.5 | 2 | 3% | 8.0% | 19% | 5.0% | 22.4% |
| 100-300 | 0.5 | 6 | 1% | 3.3% | 5% | 2.6% | 11% |
| 100-300 | 1.5 | 6 | 1% | 4.7% | 5% | 3.3% | 13% |

Table 6: Retailers with nonidentical means and two levels of coefficient of variation.

From the table we see that the chain is an efficient configuration if properly linked. The best chain linking is 3% higher than the cost of the most efficient configuration in settings with high demand uncertainty and low transshipment costs. We also see that the average linking difference for a chain configuration is less than 3%, so the manner in which retailers are arranged in a chain matters little. Also, the difference between the worst chain linking and the most efficient network is on average 3% when high demand uncertainty and low transshipment cost scenarios are excluded. In such scenarios the chain, in general, is not an efficient network. Otherwise, a poorly linked chain remains an efficient transshipment network. The efficiency of the chain network is robust to nonidentical demand parameters as well as retailer arrangement. This adds a degree of flexiblity in planning chain configurations when constraints such as geographic limits on linking arise.

### 5.2.3 Correlated Demands

We study the effect of demand correlation when (*i*) all six retailers are correlated, and (*ii*) retailers are correlated in pairs and independent of all others. In both cases, the level of correlation is identical between all correlated pairs. We consider four cost/demand scenarios in our experiments, where $\gamma = \{0.5, 1.5\}$, $c_h = 2$, $c_s = 11$ and $c_t = \{2, 6\}$.

In the first case, we set the correlation $\rho$ between each pair of retailers to be -0.1, 0, 0.3 and 0.8. Typical of inventory pooling behavior, the costs of all transshipment networks increase with correlation, converging to the cost of no transshipment. The significance of network design is most important for $\rho = -0.1$ when transshipments have the largest contribution. Most importantly, correlation does not significantly affect the ranking of the configurations with respect to efficiency. This is useful in that, for a setting where all retailer demands are correlated at the same level, the choice of an efficient transshipment network is robust to the level of demand correlation.

In the second case, where retailers are correlated in pairs and independent of all others, we set $\rho = -0.8, -0.3, 0, 0.3, 0.8$ for $\gamma = 0.5$ and $\rho = -0.3, 0, 0.3, 0.8$ for $\gamma = 1.5$. We consider all possible linkings for each configuration. The optimality gap results for the best and worst linking of the chain, star and group networks for $c_t = 2$ are presented in Tables 7 and 8 for $\gamma = 0.5$ and $\gamma = 1.5$, respectively. The results for $c_t = 6$ are similar and consistent with previous results with higher transshipment costs, where the optimality gaps are not as significant.

In each of the experiments, the chain is the most efficient configuration when properly configured. From the tables, it is clear the impact of network design increases with the absolute value of $\rho$. In addition, the difference between the linking optimality gaps of the

| | Linking Optimality Gap | | | | | |
|---|---|---|---|---|---|---|
| $\rho$ | Best chain | Worst chain | Best star | Worst star | Best group | Worst group |
| -0.8 | 0.0% | 13.6% | 22.0% | 22.5% | 39.6% | 40.3% |
| -0.3 | 0.0% | 2.9% | 7.3% | 7.4% | 24.3% | 24.7% |
| 0 | 0.0% | 0.0% | 3.8% | 3.8% | 20.9% | 20.9% |
| 0.3 | 0.0% | 1.7% | 3.6% | 3.6% | 21.1% | 21.3% |
| 0.8 | 0.0% | 3.1% | 3.7% | 4.1% | 21.9% | 22.0% |

Table 7: Retailers correlated in pairs, $\gamma = 0.5$

| | Linking Optimality Gap | | | | | |
|---|---|---|---|---|---|---|
| $\rho$ | Best chain | Worst chain | Best star | Worst star | Best group | Worst group |
| -0.3 | 0.0% | 5.0% | 1.4% | 2.0% | 13.9% | 14.1% |
| 0 | 0.0% | 0.0% | 0.2% | 0.2% | 13.2% | 13.2% |
| 0.3 | 0.0% | 3.6% | 2.6% | 3.2% | 17.2% | 17.2% |
| 0.8 | 0.0% | 8.0% | 5.3% | 7.1% | 20.5% | 20.6% |

Table 8: Retailers correlated in pairs, $\gamma = 1.5$

best chain and worst chain, indicate that the arrangement of the retailers in the chain can be a significant element of network design. This is not the case for correlated pairwise demand in manufacturing flexibility, see Jordan and Graves (1995). If the correlation is negative, it is better to place correlated retailers as neighbors in the chain; if the correlation is positive, it is better to place the correlated retailers farther apart in the chain. In contrast, the arrangement of retailers in a star or group configuration has little impact on their efficiency.

# 6 Discussion

Our work is a first step in transshipment network design research. We have introduced the chain configuration to the transshipment literature and demonstrated its advantages under a variety of demand and cost parameters. Our analysis focuses on configurations with $N$ nodes and $N$ links, consistent with the structure of the chain configuration. One area of future research is to investigate the network design problem for a general number of links. Because of the number of possible configurations, an effective network evaluation method or metric would be useful.

Additionally, in numerical studies we have relaxed several of the initial assumptions and defined scenarios in which the chain is an efficient and robust transshipment network. In future research, these models can be further generalized. In particular, it is important to investigate the transshipment network design problem in which the unit transshipment cost between different locations depends on the specific pair of locations (non-homogeneous transshipment costs). Such transshipment cost structure can represent the distance between the locations, or other location-pair characteristics. As with the case of nonidentical demand distributions, the linking of the configuration must be considered and finding the linking with minimal total expected costs is non-trivial. The nodes will differ in their accessibility due to different (transshipment) costs of connected links and varying order-up-to levels. Future research can build on the framework and basic fundamental analysis in this paper to study particular cases of these more general settings.

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# References

Chou, M., C. Teo, H. Zheng. 2005. Process flexibility revisited: graph expander and food-from-the-heart. Dept. of Decision Sciences, National University of Singapore, Singapore.

Council of Logistics Management. 2004. 15th Annual State of Logistics Report.

Dong, L., N. Rudi. 2004. Who benefits from transshipment? Exogenous vs. endogenous wholesale prices. *Management Science*, **50**, 645–657.

Graves, S.C., B.T. Tomlin. 2003. Process flexibility in supply chains. *Management Science*, **49**, 907-919.

Gurumurthi S., S. Benjaafar. 2004. Modeling and analysis of flexible queuing systems. *Naval*

*Research Logistics*, **51**, 755–782.

Herer, Y.T., M. Tzur, E. Yücesan. 2002. Transshipments: an emerging inventory recourse to achieve supply chain leagility. *International Journal of Production Economics*, **80**, 201–212.

Herer, Y.T., M. Tzur, E. Yücesan. 2006. The multi-location transshipment problem. *IIE Transactions*, **38(3)**, 185–200.

Hopp, W.J., E. Tekin, M.P. Van Oyen. 2004. Benefits of skill-chaining in production lines with cross-trained workers. *Management Science*, **50**, 83–98.

Iravani, S.M.R., K.T. Sims, M.P. Van Oyen. 2005. Structural flexibility: A new perspective on the design of manufacturing and service operations. *Management Science*, **51**, 151–166.

Iravani, S.M.R., B. Kolfal, M.P. Van Oyen. 2007a. Call center labor cross-training: It's a small world after all. *Management Science*, **53**, 1102–1112.

Iravani, S.M.R., V. Krishnamurthy. 2007b. Workforce agility in repair and maintenance environments. *Manufacturing & Service Operations Management*, **9**, 168–184.

Jordan, W.C., R.P. Inman, D.E. Blumenfeld. 2004. Chained cross-training of workers for robust performance. *IIE Transactions*, **36**, 953–967.

Jordan, W.J., S.C. Graves. 1995. Principles on the benefits of manufacturing process flexibility. *Management Science*, **41**, 577-594.

Krishnan, K.S., V.R.K. Rao. 1965. Inventory control in N warehouses. *Journal of Industrial Engineering*, **16**, 212–215.

Robinson, L.W. 1990. Optimal and approximate policies in multiperiod multilocation inventory models with transshipments. *Operations Research*, **38**, 278–295.

Sheikhzadeh, M., S. Benjaafar, D. Gupta. 1998. Machine sharing in manufacturing systems: flexibility versus chaining. *International Journal of Flexible Manufacturing Systems*,

**10**, 351–378.

Tagaras, G. 1989. Effects of pooling in two-location inventory systems. *IIE Transactions*, **21**, 250–257.

Tagaras, G., M. Cohen. 1992. Pooling in two-location inventory systems with non-negligible replenishment lead times. *Management Science*, **38**, 1067–1083.

Tagaras, G. 1999. Pooling in multi-location periodic inventory distribution systems. *Omega*, **27**, 39–59.

Tagaras, G., D. Vlachos. 2002. Effectiveness of stock transshipment under various demand distributions and nonnegligible transshipment times. *Production and Operations Management*, **11**, 183–198.

Van Mieghem, J. A., N. Rudi. 2002. Newsvendor networks: Inventory management and capacity investment with discretionary activities. *Manufacturing & Service Operations Management*, **4**, 313–335.

Yu, D.Z., S.Y. Tang, H. Shen, J. Niederhoff. 2005. On benefits of operational flexibility in a distribution network with transshipment. John M. Olin School of Business, Washington University, St. Louis, Missouri.

Yücesan, E. 2005. Private communication.

# Appendices

## A Proofs of Lemmas and Theorems

**Proof of Lemma 1**

Since only one shift is profitable, every cost-reducing transshipment incurs a cost of $c_t$ and saves $c_s + c_h$. Recall that $c_t - (c_s + c_h) \leq 0$ when the number of profitable shifts is 1. Let $p$ denote the number of units transshipped. Therefore, for a given demand realization, the final objective value is reduced by $p[c_t - (c_s + c_h)]$ when compared to the objective value when no transshipments are made. Since every cost-reducing transshipment reduces the inventory by one unit, and this is the only way to reduce inventory, minimizing inventory is equivalent to maximizing transshipments, and doing so minimizes total cost. The same argument holds for minimizing shortage. Since this holds for any one demand realization, it holds in the expectation. $\square$

**Proof of Lemma 2**

Let $z(\mathcal{K}, \mathbf{S}) = E_{\mathbf{D}}[z(\mathcal{K}, \mathbf{S}, \mathbf{d})]$ be the expected cost of the transshipment problem given $\mathcal{K}$ and $\mathbf{S}$. Herer et al. (2006) show that given $\mathcal{K}$, $z(\mathcal{K}, \mathbf{S})$ is jointly convex in $\mathbf{S} = (S_1, S_2, ..., S_N)$; therefore, the optimal solution for the values of $(S_1, S_2, ..., S_N)$ is unique. In chain and group($\ell$) configurations, the network characteristics are identical for all locations, and since the demand cost attributes are also identical, $z(\mathcal{K}, S_1, S_2, ..., S_N)$ for a given $\mathcal{K}$ is symmetric in every $S_i$.

From uniqueness and symmetry we conclude that in the optimal solution all $S_i$ values are identical. (Otherwise, suppose $S_i \neq S_j$ for some $i \neq j$. By symmetry, if we switch the values of $S_i$ and $S_j$, the solution remains optimal, which contradicts the uniqueness of the optimal solution.) $\square$

**Proof of Theorem 1**

Here, let $\mathcal{K}_c$ and $\mathcal{K}_g$ denote the unidirectional chain and group(2) configurations respectively. The order-up-to levels are identical at all locations by Lemma 2; define $S^{\mathcal{K}_g}$ to be the optimal order-up-to levels for the group configuration (i.e. $S_i = S^{\mathcal{K}_g} \;\; \forall i \in \{1..N\}$). We prove that for $\mathbf{S} = \mathbf{S}^{\mathcal{K}_\mathbf{g}}$ , the expected cost of chain is less than or equal to the expected cost of group(2), which is optimal. Clearly then, the optimal expected cost of the chain, where $\mathbf{S} = \mathbf{S}^{\mathcal{K}_\mathbf{c}}$, is lower, thus proving the theorem. If no shifts are profitable, then the expected costs of both networks are equal since no transshipments are made. We prove that the theorem holds for (i) one profitable shift and (ii) multiple profitable shifts.

*(i) One profitable shift.* When only one shift is profitable, minimizing the expected inventory yields a solution with minimum expected network cost (Lemma 1). We prove that the expected inventory in a chain is equal to the expected inventory in group(2).

For both configurations, let $I_i = S^{\mathcal{K}_g} - D_i$ be the net inventory at node $i$ before transshipment, and define $G(I_i)$ as the distribution of $I_i$. Since $S_i$ is identical for all $i$ and $D_i$ is identically distributed for all nodes, $G(I_i)$ is identical for all $i$.

Consider the chain network where node $i$ ships to node $i+1$ (except for node $N$ which ships to node 1). The positive inventory after transshipment at node $i$ is $I_i^+(chain) = min\{max(I_i + I_{i+1}, 0), max(I_i, 0)\}$. The expected inventory at node $i$ is:

$$E[I_i^+(chain)] = E\left[min\{max(I_i + I_{i+1}, 0), max(I_i, 0)\}\right] = \int_{-\infty}^{0} \int_{-I_{i+1}}^{\infty} (I_i + I_{i+1})\, dG(I_i)dG(I_{i+1}) + \int_{0}^{\infty} \int_{0}^{\infty} (I_i) dG(I_i)dG(I_{i+1}) \quad \text{(A-1)}$$

Let $I^+(chain)$ be the positive inventory after transshipment of the entire chain system. Given that only one shift is profitable, a node will not send and receive simultaneously. Since all $N$ nodes are identical and demand at each node is independent of all other nodes, the expected inventory of the entire chain system is:

$$E[I^+(chain)] = N\left(\int_{-\infty}^{0} \int_{-I_{i+1}}^{\infty} (I_i + I_{i+1})\, dG(I_i)dG(I_{i+1}) + \int_{0}^{\infty} \int_{0}^{\infty} (I_i) dG(I_i)dG(I_{i+1})\right) \quad \text{(A-2)}$$

In the group(2) network, node $i$ ships to and receives from node $i'$ when both $i$ and $i'$ belong to the same group. The inventory at a pair of such nodes $i$ and $i'$ is $max(I_i + I_{i'}, 0)$; therefore, the expected inventory in one group of the group(2) system is:

$$E[I^+_{i,i'}(group(2))] = E\left[max(I_i + I_{i'}, 0)\right] =$$
$$\int_0^\infty \int_0^\infty (I_i + I_{i'})\, dG(I_i)dG(I_{i'}) + \int_{-\infty}^0 \int_{-I_{i'}}^\infty (I_i + I_{i'})\, dG(I_{i'})dG(I_i) +$$
$$\int_{-\infty}^0 \int_{-I_i}^\infty (I_i + I_{i'})\, dG(I_i)dG(I_{i'}) =$$
$$2\left(\int_{-\infty}^0 \int_{-I_i}^\infty (I_i + I_{i'})dG(I_{i'})dG(I_i) + \int_0^\infty \int_0^\infty (I_i)dG(I_i)dG(I_{i'})\right) \quad \text{(A-3)}$$

Since all groups are identical and demand at each node is independent of all other nodes, the expected inventory of the entire group(2) system is:

$$E[I^+(group(2))] =$$
$$2M\left(\int_{-\infty}^0 \int_{-I_i}^\infty (I_i + I_{i'})dG(I_{i'})dG(I_i) + \int_0^\infty \int_0^\infty (I_i)dG(I_i)dG(I_{i'})\right) \quad \text{(A-4)}$$

Since the distributions for $I_i$ and $I_{i'}$ are the same and $N = 2M$, (A-4) is the same as (A-2). Therefore, given cost parameters such that one shift is profitable, the expected costs of these two networks are identical.

*(ii) Multiple profitable shifts.* If cost parameters allow for more than one profitable shift, the chain network is always as good as or superior to the group(2) network, since for a chain network, allowing more than one shift expands the options available to reduce costs. Alternatively, multiple shifts are not feasible in group(2) networks. Therefore, if multiple shifts are profitable, the expected cost of a chain network is less than or equal to the expected cost of the group(2) network. □

**Proof of Lemma 3**

We first describe how an optimal transshipment solution may be obtained. Since only one shift is profitable, no units are transshipped between nodes 1 and 3 (through node 2). When node 2 has a surplus (shortage) we assume, without loss of optimality, that possible transshipments from (to) node 1 are first exhausted, and only subsequently transshipments from (to) node 3 occur.

Consider the following eight cases, covering all possible realizations of net inventory/shortage levels of the nodes after demand realization, before transshipments. For each case we note in Table A-1 whether $Y_i$, the net transshipment flow into node $i$, is positive or negative (both in a weak sense), or zero.

| **Case** | **Realization** | $Y_1$ | $Y_2$ | $Y_3$ |
|---|---|---|---|---|
| 1 | All nodes are short | 0 | 0 | 0 |
| 2 | No nodes are short | 0 | 0 | 0 |
| 3 | Only node 1 is short | $\geq 0$ | $\leq 0$ | 0 |
| 4 | Only node 2 is short | $\leq 0$ | $\geq 0$ | $\leq 0$ |
| 5 | Only node 3 is short | 0 | $\leq 0$ | $\geq 0$ |
| 6 | Only node 1 is not short | $\leq 0$ | $\geq 0$ | 0 |
| 7 | Only node 2 is not short | $\geq 0$ | $\leq 0$ | $\geq 0$ |
| 8 | Only node 3 is not short | 0 | $\geq 0$ | $\leq 0$ |

Table A-1: Transshipment flows for eight exhaustive cases.

From Table A-1 we observe that the levels of $Y_1$ and $Y_3$ are either positive together, negative together, or one is zero, hence we conclude that $E(Y_1 \cdot Y_3) \geq 0$. Further, we claim that $E(Y_1) = 0$ since according to the above solution procedure the net flow out of node 1 may be obtained by ignoring node 3, in which case nodes 1 and 2 are expected to send the same amount of flow to each other. From these results, we conclude that $E(Y_1 \cdot Y_3) \geq E(Y_1) \cdot E(Y_3)$ and therefore the correlation between the transshipment flow into nodes 1 and 3 is positive; that is, $\sigma_{Y_1 Y_2} \geq 0$.

Consider the random variables representing the ending inventory/shortage at nodes 1 and 3 after demand realization and after transshipments, i.e., the random variables $I_1 + Y_1$ and $I_3 + Y_3$. The covariance between these two variables follows the equation: $\sigma_{(I_1+Y_1)(I_3+Y_3)} = \sigma_{I_1 I_3} + \sigma_{Y_1 I_3} + \sigma_{I_1 Y_3} + \sigma_{Y_1 Y_3}$. The first term, $\sigma_{I_1 I_3} = 0$ because $I_1$ and $I_3$ are independent; the second term, $\sigma_{Y_1 I_3} = 0$ because, according to our solution procedure, $Y_1$ depends on $I_1$ and $I_2$ only. For the third term, $I_1$ and $Y_3$ are related through node 2, as in cases 4 and 7 of Table A-1. In case 4 we note that the higher $I_1$, the less node 3 needs to transship to node 2, therefore the higher $Y_3$. In case 7, we note that the higher $I_1$, the less node 2 needs to transship to it, therefore node 2 may transship more to node 3, so $Y_3$ is higher. Thus $\sigma_{I_1 Y_3} \geq 0$. Finally, from above, $\sigma_{Y_1 Y_2} \geq 0$ so we conclude that $\sigma_{(I_1+Y_1)(I_3+Y_3)} \geq 0$. □

**Proof of Lemma 4**

Number the nodes in the group 1, 2 and 3. Without loss of generality, in the first step, units are transshipped on (1,2) and (2,3) only. In the second step, units may be transshipped on (1,3). Consider again the solution procedure in the proof of Lemma 3 as well as the eight cases in Table A-1. In cases 1, 2, 4 and 7 the solution using only links (1,2) and (2,3) are optimal since link (1,3) is never used for transshipment.

For case 3, let $I_i^*$ ($I_i^{**}$) be the inventory/shortage level at node $i$ after demand realization and after transshipment of the first step (first and second steps). Therefore, $I_1^* = min(I_1 + I_2, 0)$ and $I_2^* = max(I_1 + I_2, 0)$, resulting in two sub-cases:

**Case $\boldsymbol{I_1 + I_2 < 0}$.** Therefore, $I_1^* = I_1 + I_2$ and $I_2^* = 0$. In the second step, node 3 may transship to node 1, therefore: $I_1^{**} = min(I_1^* + I_3, 0)$ and $I_3^{**} = max(I_1^* + I_3, 0)$. Thus, letting $\hat{I} = I_1 + I_2 + I_3$, if $\hat{I} < 0$ the resulting inventory/shortage levels at the nodes are: $I_1^{**} = \hat{I}$ and $I_2^{**} = I_3^{**} = 0$; while, if $\hat{I} \geq 0$ then $I_1^{**} = I_2^{**} = 0$ and $I_3^{**} = \hat{I}$. In both cases, the solution is clearly optimal.

**Case $\boldsymbol{I_1 + I_2 \geq 0}$.** Therefore, $I_1^* = 0$ and $I_2^* = I_1 + I_2$. In this case we also have $I_3 \geq 0$ and therefore no transshipments will occur between nodes 1 and 3 in the second step. The solution of the first step is clearly optimal.

A similar analysis applies to cases 5, 6 and 8, which concludes the proof. □

**Proof of Theorem 2**

By Lemma 2 the optimal order-up-to level for the group configuration is equal at all locations. Let $\mathbf{S} = \mathbf{S}^{\mathcal{K}_\mathbf{g}}$. Consider first the case where in both the group and the chain configurations, the order-up-to level is set to $\mathbf{S}$ and the cost parameters are such that there is only one profitable shift. (At the end of the proof we relax those assumptions.) We analyze the solution obtained for the transshipment quantities.

Let $\mathcal{K}_1 = \{(1,2),(2,3),(4,5),(5,6)\}$ be the set of transshipment links that are identical to both the group and the chain configurations, and let $\mathcal{K}_{2g} = \{(1,3),(4,6)\}$ and $\mathcal{K}_{2c} = \{(1,6),(3,4)\}$ be the set of transshipment links that are unique (among these two configurations) to the group and the chain configurations, respectively.

Now, for any demand realization and for each configuration, assume that the solution is obtained in three steps, as follows:

**Step 1.** A solution is obtained while using transshipment links in $\mathcal{K}_1$ only, and denoting the resulting transshipment quantities by $\mathcal{X}_1$.

**Step 2.** While fixing the transshipment quantities $\mathcal{X}_1$, add the transshipment links $\mathcal{K}_{2g}$ and $\mathcal{K}_{2c}$ and obtain (possibly) additional transshipment units, $\mathcal{X}_{2g}$ and $\mathcal{X}_{2c}$ for the group and chain configurations, respectively.

**Step 3.** Attempt to improve the solutions $\mathcal{X}_1 \cup \mathcal{X}_{2g}$ and $\mathcal{X}_1 \cup \mathcal{X}_{2c}$ while using the set of transshipment links $\mathcal{K}_1 \cup \mathcal{K}_{2g}$ and $\mathcal{K}_1 \cup \mathcal{K}_{2g}$, respectively.

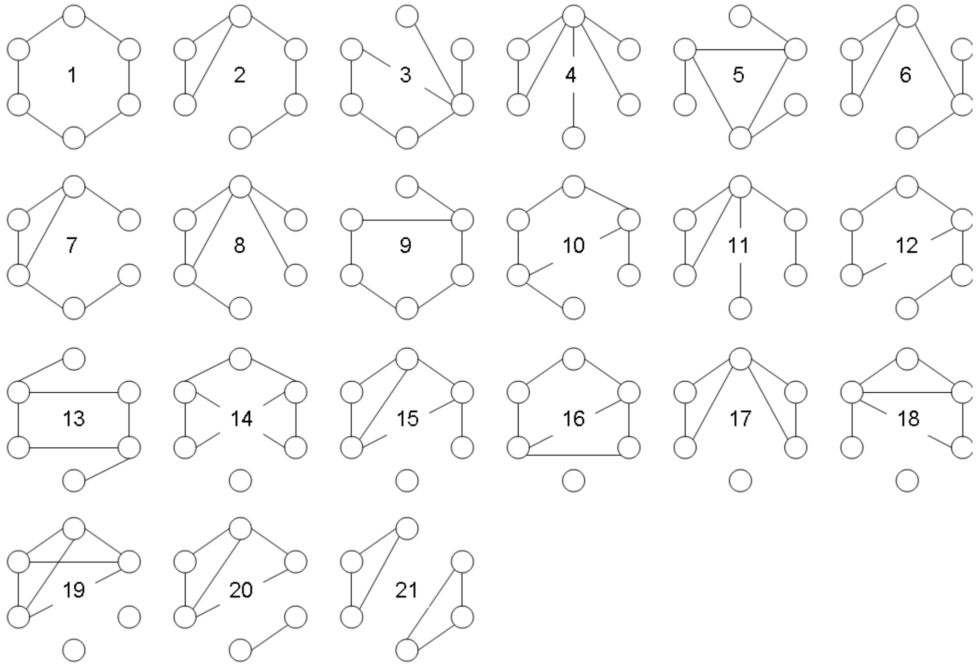
Note that in step 1, $\mathcal{K}_1$ is identical to both configurations, and therefore the solution obtained in this step is identical as well. From Lemma 3, the ending inventory/shortage levels after step 1 at nodes 1 and 3 (and similarly 4 and 6) in the group are positively correlated. Therefore, in step 2 we claim that the expected number of units transshipped by $\mathcal{X}_{2g}$ is lower than the expected number of units transshipped by $\mathcal{X}_{2c}$, since the inventory/shortage levels between nodes 1 and 6, and 3 and 4, in the chain are independent. Since every unit transshipped is associated with an identical saving, the chain configuration has a lower cost at the conclusion of Step 2. Finally, by Lemma 4, the solution of the group configuration at the conclusion of Step 2 is optimal and cannot be improved. On the other hand, the solution of the chain configuration at the conclusion of Step 2 may be improved in Step 3. Thus, the chain configuration has a lower cost at the conclusion of Step 3 as well.

This concludes the proof for $\mathbf{S} = \mathbf{S}^{\mathcal{K}_\mathbf{g}}$ and only one profitable shift. If we relax these two assumptions, it may improve the solution of the chain configuration only, and thus the proof is complete. $\square$

**<u>Proof of Theorem 3</u>**

Number the nodes in the chain network such that the bidirectional links are of the form $(i, i+1)$ for $i = 1, ..., N-1$ and $(N, 1)$. Similarly, number the nodes in the group(3) network such that each group $i$ consists of nodes $3i-2, 3i-1$, and $3i$ for $i = 1, ..., M$. The proof follows the same steps and arguments as in the proof of Theorem 2, with the following redefined sets: $\mathcal{K}_1 = \{(1,2),(2,3),(4,5),(5,6),...,(3i-2,3i-1),(3i-1,3i),...,(3M-2,3M-1),(3M-1,3M)\}$, $\mathcal{K}_{2g} = \{(1,3),(4,6),...,(3i-2,3i),...,(3M-2,3M)\}$ and $\mathcal{K}_{2c} = \{(3,4),(6,7),...,(3i,3i+1),...,(3(M-1),3(M-1)+1),(3M,1)\}$. $\square$

# B All 21 6 Location - 6 Link Network Configurations




# C Efficiency results: 12 and 18 node networks

Efficiency of the chain network for $N = 12$

| | 25 randomly generated configurations | | | | | Star configuration | | | |
|---|---|---|---|---|---|---|---|---|---|
| | # of pairwise | Chain is most | $\Delta_f(\gamma,\cdot,\cdot)$ | | Maximum | # of pairwise | Chain is most | $\Delta_f(\gamma,\cdot,\cdot)$ | |
| $\gamma$ | comparisons | efficient(%) | Avg. | Max. | difference | comparisons | efficient(%) | Avg. | Max. |
| 0.25 | 450 | 100% | - | - | 22.1% | 18 | 100% | - | - |
| 0.5 | 450 | 100% | - | - | 23.0% | 18 | 100% | - | - |
| 1 | 450 | 100% | - | - | 18.5% | 18 | 88.9% | 2.1% | 2.3% |
| 1.5 | 450 | 98.7% | 1.0% | 2.0% | 13.0% | 18 | 72.2% | 7.6% | 13.4% |
| 2 | 450 | 88.7% | 1.5% | 6.2% | 11.0% | 18 | 61.1% | 11.0% | 20.7% |
| Total | 2250 | 97.5% | 1.4% | 6.2% | 23.0% | 90 | 84.4% | 8.5% | 20.7% |

Efficiency of the chain network for $N = 18$

| | 25 randomly generated configurations | | | | | Star configuration | | | |
|---|---|---|---|---|---|---|---|---|---|
| | # of pairwise | Chain is most | $\Delta_f(\gamma,\cdot,\cdot)$ | | Maximum | # of pairwise | Chain is most | $\Delta_f(\gamma,\cdot,\cdot)$ | |
| $\gamma$ | comparisons | efficient(%) | Avg. | Max. | difference | comparisons | efficient(%) | Avg. | Max. |
| 0.25 | 450 | 100% | - | - | 32.6% | 18 | 100% | - | - |
| 0.5 | 450 | 100% | - | - | 30.7% | 18 | 100% | - | - |
| 1 | 450 | 100% | - | - | 21.4% | 18 | 83.3% | 4.6% | 6.4% |
| 1.5 | 450 | 99.3% | 0.7% | 1.2% | 14.8% | 18 | 66.7% | 9.2% | 18.6% |
| 2 | 450 | 90.2% | 1.3% | 4.6% | 11.4% | 18 | 61.1% | 14.1% | 26.4% |
| Total | 2250 | 97.9% | 1.2% | 4.6% | 32.6% | 90 | 82.2% | 10.5% | 26.4% |